J Glob Optim (2009) 44:481–492
DOI 10.1007/s10898-008-9345-3

Some regularities for parametric equilibrium problems

M. Bogdan · J. Kolumbán

Received: 19 November 2007 / Accepted: 14 August 2008 / Published online: 11 September 2008

Abstract In this paper we aim to study a family of equilibrium problems governed by pseudomonotone maps depending on a parameter and the behavior of their solutions. The main result gives sufficient conditions for closedness of the solution map defined on the set of parameters.

Keywords Equilibrium problems · Mosco convergence · Closed set-valued maps · Pseudomonotonicity

Mathematics Subject Classification (2000) 49J93D

1 Introduction

Over the last decades, there has been an increasing interest in studying parametric equilibrium problems and their particular cases, such as variational inequalities, optimization problems, mini–max point problems, and Nash equilibrium problems. As a result of changes in the problem data, the behavior of the solutions of such problems is always of concern. Our aim is to investigate three regularity properties of these solutions. To be more precise, we formulate the parametric equilibrium problem.

In this paper (X, σ) is a Hausdorff topological space and P (the set of parameters) is another Hausdorff topological space. For a given $p \in P$ we consider the following equilibrium problem:

$(EP)_p$ Find an element $a_p \in D_p$ such that

$$f_p(a_p, b) \geq 0, \quad \forall b \in D_p, \tag{1}$$

M. Bogdan (✉)
Petru Maior University, Târgu-Mureş, Romania
e-mail: bgdmarcel@netscape.net

J. Kolumbán
Department of Analysis and Optimization, Babeş-Bolyai University, Cluj-Napoca, Romania
e-mail: kolumban@math.ubbcluj.ro

where D_p is a nonempty subset of X and $f_p(\cdot,\cdot) : X \times X \rightarrow \mathbb{R}$ is a given function.

Denote by $S(p)$ the set of the solutions for a fixed p. Suppose that $S(p) \neq \emptyset$, for all $p \in P$. (For sufficient conditions for the existence of solutions see, e.g. [10,16,24,26]).

There are several concepts of regularity for parametric equilibrium problems. We intend to study the following three of them:

(i) For $p_0 \in P$ fixed and for each net of elements $(p_i, a_{p_i}) \in Graph\, S$, $(i \in I)$ $p_i \rightarrow p_0$ and $a_{p_i} \stackrel{\sigma}{\rightarrow} a_{p_0}$ imply $(p_0, a_{p_0}) \in Graph\, S$, (*closedness of* S *at* p_0);

(ii) For $p_0 \in P$ fixed with $S(p_0) = \{a_{p_0}\}$ and for each net of elements $(p_i, a_{p_i}) \in Graph\, S$, $(i \in I)$ such that $p_i \rightarrow p_0$, one has $a_{p_i} \stackrel{\sigma}{\rightarrow} a_{p_0}$ (*Hadamard well-posedness of* $(EP)_p$ *at* p_0);

(iii) For $p_0 \in P$ fixed with $S(p_0) \neq \emptyset$ and for each net of elements $(p_i, a_{p_i}) \in Graph\, S$, $(i \in I)$ such that $p_i \rightarrow p_0$, (a_{p_i}) must have a subnet $\sigma-$converging to an element of $S(p_0)$ (*generalized Hadamard well-posed of* $(EP)_p$ *at* p_0).

There are numerous results concerning these properties. In the context of variational inequalities governed by monotone hemicontinuous operators, Mosco obtained closedness of the solution map for parametric variational inequalities (see [38], Theorem A and Theorem B). Gwinner [19] and Lignola and Morgan [30–32] established closedness of the solution map for parametric variational inequalities under more general monotoniticity assumptions. Li et al. [29] and more recently Khanh and Luu [21] established closedness of the solution map for parametric quasi-variational inequalities. The closedness of the solution map for parametric variational inequalities governed by operators of Višik type was proved in [13].

Many researchers investigated well-posedness for some specific problems (see [16,35] and references therein). Relationships between different concepts of well-posedness for optimization and Nash equilibria, such as Tykhonov well-posedness, Hadamard well-posedness, and Levitin—Polyak well-posedness have been given in [11,36,37,39,42,43].

Upper semi-continuity in the first variable of the function $f_p(\cdot,\cdot)$ is supposed in most of the papers mentioned above. This condition is motivated by restrictions occurring in economy. Even if it looks natural, in some important particular cases it is not satisfied. This happens, for instance, in the case of variational inequalities governed by differential operators (see Sect. 4 below). To compensate for the lack of upper semi-continuity, Brézis [14] introduced the notion of topological pseudomonotonicity (which is a kind of conditioned upper semi-continuity) in the context of variational inequalities.

The study of regularization methods for perturbed variational inequalities with single-valued pseudomonotone operators was initiated in [33]. Penalty methods for variational inequalities were given in [18] and [1]. Regularization and penalization for variational inequalities with set-valued pseudomonotone operators are unified and sufficient conditions for generalized Hadamard well-posedness are given in [34]. Some results on regularized equilibrium problems related to generalized Hadamard well-posedness were obtained in [20] and [15].

The present paper offers sufficient conditions for the mentioned regularities. To achieve this we used, among other instruments, a generalization of the topological pseudomonotonicity given in [8]. This notion allowed us to derive, from the obtained results, some of the known statements in the specialized literature (see Sects. 4, 5).

The main results related to closedness are contained in Theorem 1 and Theorem 2 (see Sect. 3). From them Hadamard well-posedness and generalized Hadamard well-posedness in Corollary 1 and Corollary 3 can easily be deduced. Our results generalize most of the statements mentioned above about properties (i), (ii), and (iii), for real valued parametric bifunctions.

The paper is organized as follows. In Sect. 2 we give the definition of the topological pseudomonotonicity of bifunction and of Mosco convergence of parametric domains from a space with two topologies. We also present some known results we need in the subsequent sections. Section 3 contains our results about closedness of the solution map (Theorem 1) and generalized Hadamard well-posedness of problem $(EP)_p$ (Corollary 1). Some examples are given in order to illustrate the relevance of the condition (**C**) about the dependence of f_p on the parameter p, imposed in Theorem 1. In Sect. 4 parametric variational inequalities governed by nonlinear differential operators of Višik type are treated and the closedness of the solution map is deduced. In Sect. 5 we discuss the case of constant domains, i.e. D_p does not depent on p. In this case, the condition (**C**) can be weakened. As applications, some known results on Walras equilibrium points, Ky Fan equilibrium points, and Hadamard well-posedness of problem $(EP)_p$ in two particular cases considered in [11] are deduced.

Besides (i), (ii), (iii) there are several other regularity properties, such as: upper semi-continuity of the solution map [2,4,5,22,23], continuity of the solution map [17], Hölder estimations of the solutions [3,7,8,12], derivability with respect to the parameter of the solution map, etc., however these do not make the subject of the present paper.

2 Definitions and auxiliary results

We will use the following generalization of topological pseudomonotonicity introduced by Brézis in [14].

Definition 1 [8, p. 410] A function $g : X \times X \to \mathbb{R}$ is said to be topologically pseudomonotone if for each net $(a_i)_{i\in I} \subset X$ with $a_i \xrightarrow{\sigma} a$ in X, $\liminf_i g(a_i, a) \geq 0$ imply

$$\limsup_i g(a_i, b) \leq g(a, b), \quad \text{for all } b \in X.$$

For the parametric domains in $(EP)_p$ we shall use the following type of convergence, which is a slight generalization of Mosco's convergence in [38]. In the following, besides the topology σ, we also consider a stronger topology τ on X. Hence, when X is a normed space, σ can be chosen as the weak topology and τ as the strong topology on X (see Sect. 4).

Definition 2 Let D_p be subsets of X for all $p \in P$. The sets D_p converge to D_{p_0} (and write $D_p \xrightarrow{M} D_{p_0}$) as $p \to p_0$ if:

(a) for every net $(a_{p_i})_{i\in I}$ with $a_{p_i} \in D_{p_i}$, $p_i \to p_0$ and $a_{p_i} \xrightarrow{\sigma} a$ imply $a \in D_{p_0}$;

(b) for every $a \in D_{p_0}$, there exist $a_p \in D_p$ such that $a_p \xrightarrow{\tau} a$ as $p \to p_0$.

If X is a normed space and $\sigma = \tau =$ norm topology, $D_p \xrightarrow{M} D_{p_0}$ amounts to saying that the sets D_p converge to D_{p_0} in the Painlevé-Kuratowski sense as $p \to p_0$. A comparison of the Mosco convergence with the Hausdorff convergence, for example, is given in Lemma 1.1 in [38].

Let us recall some other classical definitions from set-valued analysis. Let X, Y be Hausdorff topological spaces. The set-valued map $T : Y \to 2^X$ is said to be upper semi-continuous at $y_0 \in dom\, T := \{y \in Y \mid T(y) \neq \emptyset\}$ if, for each neighborhood V of $T(y_0)$, there exists a neighborhood U of y_0 such that $T(U) \subset V$. The map T is said to be closed at $y \in dom\,T$ if, for each net $(y_i)_{i\in I}$ in $dom\,T$, $y_i \to y$ and each net $(x_i)_{i\in I}$, $x_i \in T(y_i)$ with $x_i \to x$, one has $x \in T(y)$. The map T is said to be closed if it is closed at all $y \in dom\,T$ or equivalently if its graph $Graph\,T = \{(y,x) \in Y \times X \mid x \in T(y)\}$ is closed. Closedness and upper semi-continuity of a set-valued map are closely related as shown in the following result.

Proposition 1 ([9], Proposition 1.4.8, 1.4.9) *Let X, Y be Hausdorff topological spaces.*

(a) If $T : Y \to 2^X$ has closed values and is upper semi-continuous, then T is closed;
(b) If X is compact and T is closed at $y \in Y$, then T is upper semi-continuous at $y \in Y$.

The relationship between upper semi-continuity and Hadamard well-posedness is given in the following statement.

Proposition 2 ([42], Theorem 2.2) *If the solution map $S : P \to 2^X$ is upper semi-continuous at $p_0 \in P$ and $S(p_0)$ is compact, then $(EP)_p$ is generalized Hadamard well-posed at p_0. Furthermore, if $S(p_0) = \{a_{p_0}\}$ (a singleton), then $(EP)_p$ is Hadamard well-posed at p_0.*

3 Closedness of the solution mapping

Let us return to the problem $(EP)_p$ stated in Introduction.

In order to state our main result, among others we impose the following condition at $p_0 \in P$:

(**C**) For each net of elements $(p_i, a_{p_i}) \in Graph\,S$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, and $b_{p_i} \xrightarrow{\tau} b$, then

$$\liminf_i \left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right) \le 0.$$

Remark 1 Condition (**C**) is weaker than the following one:

(**C**)$_{\mathbf{eq}}$ For each net $(p_i, a_{p_i}) \in Graph\,S$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, and $b_{p_i} \xrightarrow{\tau} b$, then

$$\lim_i \left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right) = 0.$$

Obviously, (**C**)$_{\mathbf{eq}}$ applies if X is a normed space, σ and τ being the weak and strong topology on X, respectively, and

$$|f_p(a_p, b_p) - f_{p_0}(a_p, b)| \le \alpha(p)\left(\|a_p\| + \|b_p\|\right), \text{ for each } a_p, b_p \in D_p, b \in D_{p_0},$$

with α a nonnegative function continuous at p_0 and $\alpha(p_0) = 0$.

The main result of this section is the following.

Theorem 1 *Let X be a Hausdorff topological space with σ and τ the topologies as in Sect. 2. Let D_p be nonempty sets of X, $p \in P$, and let $p_0 \in P$ be fixed. Suppose that $S(p) \neq \emptyset$, for each $p \in P$, and the following conditions hold:*

(i) $D_p \xrightarrow{M} D_{p_0}$;
*(ii) $f_p(\cdot,\cdot)$ satisfies condition (**C**) at p_0;*
(iii) $f_{p_0}(\cdot,\cdot) : X \times X \to \mathbb{R}$ is topologically pseudomonotone.

Then the solution map $p \longmapsto S(p)$ is closed at p_0, i.e. for each net of elements $(p_i, a_{p_i}) \in Graph\ S$, $p_i \to p_0$ and $a_{p_i} \to a$ imply $(p_0, a) \in Graph\ S$.

Proof Let $(p_i, a_{p_i})_{i\in I}$ be a net of elements $(p_i, a_{p_i}) \in Graph\ S$ with $p_i \to p_0$ and $a_{p_i} \xrightarrow{\sigma} a$. From $i)$ we get $a \in D_{p_0}$. Moreover, there exists a net $(\bar{a}_{p_i})_{i\in I}$, $\bar{a}_{p_i} \in D_{p_i}$ such that $\bar{a}_{p_i} \xrightarrow{\tau} a$ in X. Replacing b with $\bar{a}_{p_i}$ in (1), it follows:

$$f_{p_i}(a_{p_i}, \bar{a}_{p_i}) \geq 0.$$

By condition (**C**), for $b_{p_i} := \bar{a}_{p_i}$, we have

$$\limsup_i f_{p_0}(a_{p_i}, a) \geq 0,$$

hence, there exists a subnet of $(p_i, a_{p_i})_{i\in I}$, denoted by the same indexes, such that

$$\lim_i f_{p_0}(a_{p_i}, a) = \limsup_i f_{p_0}(a_{p_i}, a).$$

Now, we apply $iii)$ for the subnet gained above to yield

$$\limsup_i f_{p_0}(a_{p_i}, b) \leq f_{p_0}(a, b), \quad \text{for all } b \in D_{p_0}. \tag{2}$$

Finally, due to (2), for $b \in D_{p_0}$, $b_{p_i} \in D_{p_i}$, $b_{p_i} \xrightarrow{\tau} b$, and condition (**C**), we obtain

$$\begin{aligned} 0 &\leq \liminf_i f_{p_i}(a_{p_i}, b_{p_i}) \leq \limsup_i f_{p_0}(a_{p_i}, b) + \liminf_i \left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right) \\ &\leq \limsup_i f_{p_0}(a_{p_i}, b) \leq f_{p_0}(a, b), \end{aligned}$$

which completes the proof. □

The closedness of S plays an important role, for instance, in shape optimization theory. In that case $(\mathbf{C})_{\mathbf{eq}}$ is satisfied by some reasonable conditions (see [25], Remark 2).

We use the following property of the solution map S.

Lemma 1 *If D_{p_0} is closed and $f_{p_0}(\cdot, \cdot) : X \times X \to \mathbb{R}$ is topologically pseudomonotone, then $S(p_0)$ is closed.*

Proof Let $a_i \in S(p_0)$ with $a_i \xrightarrow{\sigma} a$. Thus $a \in D_{p_0}$, hence

$$0 \leq \liminf_i f_{p_0}(a_i, a).$$

Using topologically pseudomotonicity we obtain

$$0 \leq \liminf_i f_{p_0}(a_i, b) \leq \limsup_i f_{p_0}(a_i, b) \leq f_{p_0}(a, b), \quad \forall\, b \in D_{p_0},$$

consequently $a \in S(p_0)$. □

As an application we state a result about well-posedness.

Corollary 1 *Let (X, σ) be a compact Hausdorff topological space and P be a Hausdorff topological space. Let D_{p_0} be a closed subset of X. If the hypotheses of Theorem 1 are satisfied, then $(EP)_p$ is generalized Hadamard well-posed at p_0. Furthermore, if $S(p_0) = \{a_{p_0}\}$ (a singleton), then $(EP)_p$ is Hadamard well-posed at p_0.*

Proof By Theorem 1 it follows that the solution map S is closed at p_0. We may use Proposition 1 (ii) to state that S is upper semi-continuous at p_0. The set $S(p_0)$ is closed by Lemma 1, therefore compact. The conclusion follows by Proposition 2. □

For all the examples in the sequel we shall consider $P = \mathbb{N} \cup \{\infty\}$, $p_0 = \infty$, (∞ means $+\infty$ from real analysis), $D_p = [0, 1]$, $p \in P$, the real functions $f_p(\cdot, \cdot) : [0, 1] \times [0, 1] \to \mathbb{R}$, and σ, τ are the natural topology on $X = [0, 1]$. On P we consider the topology induced by the metric given by $d(m, n) = |1/m - 1/n|$, $d(n, \infty) = d(\infty, n) = 1/n$, for $m, n \in \mathbb{N}$, and $d(\infty, \infty) = 0$.

As the following example shows condition (**C**) is essential for Theorem 1.

Example 1 Let $f_n(a, b) = na \cdot exp\,(1 - na) - b$, $n \in \mathbb{N}$ and $f_\infty(a, b) = a - b$.

We have $(n, 1/n) \in Graph\ S$, for each $n \in \mathbb{N}$, $S(\infty) \neq \emptyset$ but $0 \notin S(\infty)$. Hence S is not closed at ∞. In this case f_∞ is topologically pseudomonotone, but for $b_n \to b$ one has

$$\liminf_{n\to\infty} \left(f_n(1/n, b_n) - f_\infty(1/n, b)\right) = 1 > 0,$$

so (**C**) fails at ∞.

The next example shows that topologically pseudomonotonicity of the limit function f_∞ is essential for Theorem 1.

Example 2 Let $f_n(a, b) = b - |a - 1/n|$, $n \in \mathbb{N}$ and

$$f_\infty(a, b) = \begin{cases} \sin(1/a), & a \neq 0, b \in [0, 1] \\ -1, & a = 0, b \in [0, 1] \end{cases}.$$

We have $(n, 1/n) \in Graph\ S$ for each $n \in \mathbb{N}$, $S(\infty) \neq \emptyset$, but $0 \notin S(\infty)$. Hence S is not closed at ∞.

Let us check condition (**C**) at ∞. For $a_n = 1/n \in S(n)$, $n \in \mathbb{N}$, $b_n \in [0, 1]$, and $b_n \to b \in [0, 1]$, one has

$$\liminf_{n\to\infty} \left(f_n(a_n, b_n) - f_\infty(a_n, b)\right) = b - \limsup_{n\to\infty} f_\infty(1/n, b) \leq 0.$$

In this case f_∞ is not topologically pseudomonotone since for $a_n = 1/(2n\pi + \pi/2) \to 0$ one has $\liminf_{n\to\infty} f_\infty(a_n, 0) = 1 \geq 0$, but

$$\limsup_{n\to\infty} f_\infty(a_n, 0) = 1 > -1 = f_\infty(0, 0).$$

One might have $f_n \longrightarrow f_\infty$ uniformly, but condition (**C**) does not apply at ∞ as the next example yields.

Example 3 For each $n \in P$ let us consider the functions f_n given by

$$f_n(a, b) = \begin{cases} a - b + 1/n, & b \neq 0, a \in [0, 1] \\ 0, & b = 0, a \in [0, 1] \end{cases}, \ n \in \mathbb{N},$$

and

$$f_\infty(a, b) = \begin{cases} a - b, & b \neq 0, a \in [0, 1] \\ 0, & b = 0, a \in [0, 1] \end{cases}.$$

We have obviously $f_n \longrightarrow f_\infty$ (uniformly). To brake (**C**) we choose $a_n = 1$ and $b_n = 1/n$. In this case

$$\liminf_{n\to\infty} \left(f_n(a_n, b_n) - f_\infty(a_n, 0)\right) = 1.$$

4 Parametric variational inequalities

Let X be a normed space, σ the weak topology and τ the strong topology on X.

Lemma 2 *Let X^* be the dual space of X. Let $A_p : X \to X^*$ be operators having the uniform boundedness property, namely for $a_p \in X$, $(a_p)_p$ bounded, there exists a constant $c > 0$ such that $\|A_p(a_p)\| \le c$ for any $p \in P$.*

Define $f_p : X \times X \to \mathbb{R}$ by $f_p(a, b) = \langle A_p(a), b - a\rangle$.

Suppose that A_p satisfy hypothesis

(**H**) *For each net of elements $(p_i, a_{p_i}) \in Graph\ S$ such that*

$$p_i \to p_0 \text{ and } a_{p_i} \xrightarrow{\sigma} a, \text{ and for all } b \in D_{p_0}, \text{ yields}$$
$$\liminf_i \langle A_{p_i}(a_{p_i}) - A_{p_0}(a_{p_i}), b - a_{p_i}\rangle \le 0. \tag{3}$$

Then condition (**C**) *applies at p_0.*

Proof Let $b \in D_{p_0}$, $b_{p_i} \in D_{p_i}$, and $b_{p_i} \xrightarrow{\tau} b$, as $p_i \to p_0$. Since

$$\langle A_{p_i}(a_{p_i}), b_{p_i} - a_{p_i}\rangle - \langle A_{p_0}(a_{p_i}), b - a_{p_i}\rangle = \langle A_{p_i}(a_{p_i}) - A_{p_0}(a_{p_i}), b - a_{p_i}\rangle + \langle A_{p_i}(a_{p_i}), b_{p_i} - b\rangle,$$

the assertion follows from (3). □

The hypothesis (**H**) applies in some important particular case. To see this, let Ω be a bounded open set of $\mathbb{R}^N$ with Lipschitz boundary. Let $X = H^1(\Omega)$ be the usual Sobolev space. Let us consider $P = \mathbb{N} \cup \{\infty\}$, $p_0 = \infty$. Suppose that the functions $v_i^p : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ $(i = 0, \dots, N)$ have the following properties (see [40, p. 74]):

(P1) For $i = 0, \dots, N$, $v_i^p(x, \eta, \xi)$ is measurable in $x \in \mathbb{R}^N$ and continuous in $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$;

(P2) For $i = 0, \dots, N$, $|v_i^p(x, \eta, \xi)| \le c(k(x) + |\eta| + \|\xi\|_N)$ for a.e. $x \in \mathbb{R}^N$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$, with c a positive constant and k a function in $L^4_{loc}(\mathbb{R}^N)$;

(P3) $\sum_{i=1}^{N} \left(v_i^p(x, \eta, \xi) - v_i^p(x, \eta, \tilde{\xi})\right)(\xi_i - \tilde{\xi}_i) > 0$ for a.e. $x \in \mathbb{R}^N$, $\forall \eta \in \mathbb{R}$, $\forall \xi, \tilde{\xi} \in \mathbb{R}^N$, and $\xi \ne \tilde{\xi}$;

(P4) $\sum_{i=1}^{N} v_i^p(x, \eta, \xi)\xi_i \ge c_1\|\xi\|_N^2 - c_2$ and $v_0^p(x, \eta, \xi)\eta \ge c_3|\eta|^2 - c_4$, for a.e. $x \in \mathbb{R}^N$, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$, with c_1, c_2, c_3, c_4 positive constants.

For every $p \in P$ we consider the function $f_p : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$f_p(a, b) = \int_\Omega \left\{\sum_{i=1}^{N} v_i^p(x, a(x), \nabla a(x)) \cdot \partial_i (b - a)(x)\right\} dx + \int_\Omega v_0^p(x, a(x), \nabla a(x))(b - a)(x) dx.$$

Here ∂_i denotes the partial derivative with respect to the variable x_i.

By the well known result due to Leray-Lions ([40], Theorem 6.1; [28]) the functions f_p are topologically pseudomonotone, for all $p \in P$, but $f_p(\cdot, b)$ can be not upper semi-continuous.

We have shown in [13] that (**H**) applies if one has

$$\textbf{(h)} \qquad |v_i^p(x,\eta,\xi) - v_i^\infty(x,\eta,\xi)| \le \alpha(1/p)(\|\xi\|_N + |\eta| + \tilde{k}(x)),$$

for all $p \in \mathbb{N}$, $i = 0, \ldots, N$, $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}$, and a.e. $x \in \Omega$. Here, $\tilde{k}$ has nonnegative values and belongs to $L^2(\Omega)$, α is a nonnegative function, continuous at 0 and $\alpha(0) = 0$.

Corollary 2 ([13], Theorem 1) *If the functions* v_i^p $(i = 0, \ldots, N)$ *have properties* (P1), (P2), (P3), (P4), $D_p \xrightarrow{M} D_\infty$, *and* (**h**) *applies, then the solution map* S *is closed at* ∞.

5 The case of constant domains

Let us study the particular case when $D_p = X$ for all $p \in P$. In this case condition (**C**) can be weakened to:

(**C**)$'$ For each net of elements $(p_i, a_{p_i}) \in Graph\, S$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, and $b \in X$, one has

$$\liminf_i \left(f_{p_i}(a_{p_i}, b) - f_{p_0}(a_{p_i}, b)\right) \le 0.$$

In this particular case the proof of the assertion in Theorem 1 goes easier:

$$0 \le \liminf_i f_{p_i}(a_{p_i}, b) \le \limsup_i f_{p_0}(a_{p_i}, b) + \liminf_i \left(f_{p_i}(a_{p_i}, b) - f_{p_0}(a_{p_i}, b)\right)$$
$$\le \limsup_i f_{p_0}(a_{p_i}, b) \le f_{p_0}(a, b), \quad \forall\, b \in X.$$

Therefore we have the following statement:

Theorem 2 *Consider problem* $(EP)_p$ *with* $D_p = X$. *Suppose that:*

(i$'$) $f_p(\cdot,\cdot)$ *satisfies condition* (**C**$'$) *at* p_0;
(ii$''$) $f_{p_0}(\cdot,\cdot) : X \times X \to \mathbb{R}$ *is topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ *is closed at* p_0.

As Example 3 shows, in case of constant domains it can happen that (**C**$'$) applies but (**C**) does not.

Similarly to Corollary 1 we deduce the following:

Corollary 3 *Consider* $(EP)_p$ *with* $D_p = X$ *and let* X *be compact. If the hypotheses of Theorem* 2 *are satisfied then* $(EP)_p$ *is generalized Hadamard well-posed at* p_0. *Furthermore, if* $S(p_0) = \{a_{p_0}\}$ *(a singleton), then* $(EP)_p$ *is Hadamard well-posed at* p_0.

5.1 Applications

In the following we deduce from Corollary 3 four known results.

(A_1) The first one is due to Jofré and Wets on Walras equilibrium points.

The general framework is taken over from [27]. Let us denote

$$X = \left\{ a \in \mathbb{R}_+^N \mid \sum_{j=1}^N a_j = 1 \right\}$$

the price simplex.

An equilibrium price $\bar{a} \in X$, for a pure exchange economy $\mathcal{E}$, is a solution of the inclusion

$$0 \in \mathcal{S}(a).$$

Here the set-valued function $\mathcal{S} : X \to 2^{\mathbb{R}^N}$ is defined by $\mathcal{S}(a) = s(a) - \mathbb{R}^N_+$, where the function $s : \mathbb{R}^N_+ \to \mathbb{R}^N$ is the so-called excess supply function. This function s is assumed to be nonnegative and continuous on the price simplex X (see [27], Proposition 4).

The Walrasian $W : X \times X \to \mathbb{R}$ associated with the supply function s is defined by

$$W(a, b) = \langle s(a), b \rangle.$$

The bifunction W has the following properties:

(α) $\forall b \in X : W(\cdot, b)$ is continuous;
(β) $\forall a \in X : W(a, \cdot)$ is affine;
(γ) $\forall a \in X : W(a, a) \geq 0$.

Of course, the equilibrium prices are exactly the equilibrium points of W.

Corollary 4 ([27], Theorem 15) *Consider the sequence of Walrasians $(W_n)_{n\in\mathbb{N}}$ and W associated with supply functions s_n and s, respectively. Suppose that $s_n \overset{c}{\to} s$, i.e. for any $(a_n)_{n\in\mathbb{N}}$, $a_n \in X$ with $a_n \to a$ one has $s_n(a_n) \to s(a)$. Then, for each W_n $(n \in \mathbb{N})$ and W there exists at least one equilibrium point in X, denoted by $\bar{a}_n$ and $\bar{a}$, respectively. The set of cluster points of the sequence $(\bar{a}_n)_{n\in\mathbb{N}}$ is never empty, and every cluster point is an equilibrium point for the economy $\mathcal{E}$ with the supply function s.*

Proof Let $\sigma = \tau$ be the natural topology on X. The existence of the equilibrium points is obvious. Let us check condition ($\mathbf{C}'$) at ∞. For $b \in X$ and $a_n \in S(n)$, with $a_n \to a$, using $s_n \overset{c}{\to} s$, we have

$$\liminf_n [W_n(a_n, b) - W(a_n, b)] = \liminf_n [\langle s_n(a_n), b\rangle - \langle s(a_n), b\rangle] = 0.$$

The function $W(\cdot, b)$ being continuous, W is topologically pseudomonotone.

Consequently from our Corollary 3 follows the assertion. □

(A_2) The second application is related to a recent result contained in [43].

Let X be a compact metric space. Let M be the set of all functions $f : X \times X \to \mathbb{R}$ such that $a \mapsto f(a, b)$ is upper semi-continuous for each $b \in X$ and $\sup_{(a,b)\in X\times X} |f(a, b)| < +\infty$.

For each $f, g \in M$ define the distance of uniformly convergence:

$$\rho(f, g) = \sup_{(a,b)\in X\times X} |f(a, b) - g(a, b)|.$$

Then (M, ρ) is a complete metric space. Consider the set of parameters

$$P = \{f \in M : \text{ there exists } a^* \in X \text{ such that } f(a^*, b) \geq 0, \text{ for all } b \in X\}.$$

Given $f \in P$, such an a^* is called an equilibrium point of f. (In [41] such an a^* is called a Ky Fan point of $-f$.) Denote by $S(f)$ the set of equilibrium points of f. Then $f \longmapsto S(f)$ defines a set-valued map from P to X. Let $f_n, g \in P$, $(n \in \mathbb{N})$. We shall write $f_n \overset{\rho}{\longrightarrow} g$ if f_n converges to g with respect to ρ.

Corollary 5 ([43], Theorem 3.1) *The equilibrium problem is generalized Hadamard well-posed at every $f \in P$, i.e. for any sequence $(f_n) \subset P$ with $f_n \overset{\rho}{\longrightarrow} f$ and any $a_n \in S(f_n)$, (a_n) must have a subsequence converging to an element in $S(f)$.*
Suppose furthermore that $S(f) = \{a_0\}$. Then the equilibrium problem is Hadamard well-posed at f, i.e. for any (f_n) with $f_n \overset{\rho}{\longrightarrow} f$ and any $a_n \in S(f_n)$, there must be $a_n \to a_0$.

Proof Let $f, f_n \in P$ such that $f_n \xrightarrow{\rho} f$. Condition $(\mathbf{C}')$ applies at ∞ obviously. Since $f \in M$, f is upper semi-continuous in its first variable, hence it is topologically pseudomonotone. The conclusion follows by Corollary 3. □

(A_3) Let $P = [0, +\infty)$, (X, d) be a metric space, and $f_0 : X \times X \to \mathbb{R}$ be a function with $f_0(a, a) \geq 0$, for every $a \in X$. Let us consider the following expressions for the function f_p:

(1) $f_p(a, b) = f_0(a, b) + p$;
(2) $f_p(a, b) = f_0(a, b) + pd(a, b)$.

Let σ and τ be the same topology induced by d. Obviously, in both cases 1) and 2), one has $f_p \xrightarrow{\rho} f_0$ uniformly on bounded subsets, hence $(\mathbf{C}')$ apply at $p_0 = 0$.

If f_0 is upper semi-continuous in its first variable, then so is f_p. In this case, if X is compact, we obviously have $S(p) \neq \emptyset$ for all $p \in P$.

Therefore, we have:

Corollary 6 ([11], Corollary 3) *Let (X, d) be a compact metric space, and let $f_0 : X \times X \to \mathbb{R}$ be such that $f_0(a, a) \geq 0$, for every $a \in X$. Suppose that $f_0(\cdot, b)$ is upper semi-continuous for every $b \in X$. Then for f_p defined by* 1) *or* 2)*, problem $(EP)_p$ is Hadamard well-posed at* 0.

Let $P = \mathbb{N} \cup \{\infty\}$. Observe that condition $(\mathbf{C}')$ apply at $p_0 = \infty$ while $f_n \xrightarrow{\rho} f_\infty$ (uniformly), but the reverse implication does not hold, as the following example shows.

Example 4 Let $f_n, f_\infty : [0, 1] \to \mathbb{R}$, $n \in \mathbb{N}$. Let $f_n(a, b) = \big(na \cdot exp\,(-na) - 1/e\big)(b - a)$, and $f_\infty(a, b) = 0$, for each $a, b \in [0, 1]$. Obviously, f_n does not converge even punctually to f_∞ on $X = [0, 1]$. To verify $(\mathbf{C}')$ let $a_n \in S(n)$, thus $a_n = 1/n$. For $b \in [0, 1]$ we have $\liminf_{n\to\infty} \big(f_n(a_n, b) - f_\infty(a_n, b)\big) = 0$, so $(\mathbf{C}')$ applies.

(A_4) Example 4 also provides a case when Theorem 2 applies but Theorem 2.1 in [21] does not.

For this let $X = [0, 1]$ with the natural topology, $P = \mathbb{N} \cup \{\infty\}$, and $C(a) = \mathbb{R}_+$, $g_n(a) = a$, for each $n \in P$. Let $t_n : X \to \mathbb{R}$, $n \in P$, and suppose that t_∞ is continuous. In this case the parametric vector quasi-variational inequality problem $(PQVI)_n$ studied in [21] collapses into the following parametric equilibrium problem:

Find an element $a_n \in D$ such that

$$f_n(a_n, b) \geq 0, \quad \forall b \in X,$$

with $f_n(a, b) = t_n(a) \cdot (b - a)$.

From Theorem 2.1 in [21] it follows that, if

$$\begin{aligned} &\forall a_n \to a, \forall y_n \to y_0, \exists\, t_{n_k}(a_{n_k}),\ \text{(a subsequence of } t_n(a_n)),\\ &\quad \text{such that } t_{n_k}(a_{n_k}) \cdot y_{n_k} \to t_\infty(a) \cdot y_0, \end{aligned} \tag{4}$$

then the solution map S is closed at $p_0 = \infty$.

We claim that the requirement (4) implies the hypotheses of our Theorem 1, but the reverse implication is not true.

Indeed, $(\mathbf{C})$ applies at ∞ since, if $a_n \in X$, $a_n \to a$, $b_n \to b$, then

$$\begin{aligned} &\liminf_{n\to\infty}[t_n(a_n) \cdot (b_n - a_n) - t_\infty(a_n) \cdot (b - a_n)]\\ &\leq \lim_{k\to\infty} [t_{n_k}(a_{n_k}) \cdot (b_{n_k} - a_{n_k}) - t_\infty(a_{n_k}) \cdot (b - a_{n_k})] = 0. \end{aligned}$$

Of course, f_∞ is topologically pseudomonotone. Therefore, if (4) holds then Theorem 2 applies.

On the other hand, if we consider the functions defined by $t_n(a) = na \cdot exp\,(-na) - 1/e$ and $t_\infty(a) = 0$, and choose $a_n = a$, $b_n = b > a$ in X, then

$$\lim_{n\to\infty} t_n(a_n) \cdot (b_n - a_n) = -1/e \cdot (b - a) < 0 = t_\infty(a) \cdot (b - a),$$

hence condition (4) fails.

Acknowledgements The authors are indebted to the referee for careful reading, for valuable and constructive remarks.

References

1. Alber, Y.I.: The penalty method for variational inequalities with nonsmooth unbounded operators in Banach space. Numer. Funct. Anal. Optim. **16**, 1111–1125 (1995)
2. Anh, L.Q., Khanh, P.Q.: Semicontinuity of the solution set of parametric multivalued vector quasiequilibrium problems. J. Math. Anal. Appl. **294**, 699–711 (2004)
3. Anh, L.Q., Khanh, P.Q.: On the Hölder continuity of solutions to parametric multivalued vector equilibrium problems. J. Math. Anal. Appl. **321**, 308–315 (2006)
4. Anh, L.Q., Khanh, P.Q.: On the stability of the solution sets of general multivalued vector quasiequilibrium problems. J. Opt. Theory Appl. **135**(2), 271–284 (2007)
5. Anh, L.Q., Khanh, P.Q.: Various kinds of semicontinuity and the solution sets of parametric multivalued symmetric vector quasiequilibrium problems. J. Glob. Optim. **41**, 539–558 (2008)
6. Anh, L.Q., Khanh, P.Q.: Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces. J. Glob. Optim. doi:10.1007/s10898-006-9062-8
7. Anh, L.Q., Khanh, P.Q.: Sensitivity analysis for multivalued quasiequilibrium problems in metric spaces: Hölder continuity of solutions. J Glob. Optim. doi:10.1007/s10898-007-9268-4
8. Aubin, J.P.: Mathematical Methods of Game and Economic Theory. North Holland, Amsterdam (1982)
9. Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston (1990)
10. Avriel, M., Diewert, W.E., Schaible, S., Zang, I.: Generalized Concavity. Plenum Press, New York (1988)
11. Bianchi, M., Kassay, G., Pini, R.: Well-Posed Equilibrium Problems. Contributi di ricerca in Econometrica e Matematica, n. 85-ottobre (2006)
12. Bianchi, M., Pini, R.: Sensitivity for parametric vector equilibria. Optimization **55**, 221–230 (2006)
13. Bogdan, M., Kolumbán, J.: On Nonlinear Parametric Variational Inequalities. Nonlinear Anal. **67**, 2272–2282 (2007)
14. Brézis, H.: Équations et inéquations non linéaires en dualité. Annales de l'Institut Fourier (Grenoble) **18**(1), 115–175 (1965)
15. Chadli, O., Schaible, S., Yao, J.-C.: Regularized equilibrium problems with application to noncoercive hemivariational inequalities. J. Opt. Theory Appl. **121**(3), 571–596 (2004)
16. Dontchev, A.L., Zolezzi, T.: Well-posed Optimization Problems, Lecture notes in Mathematics, 1543. Springer-Verlag, Berlin (1993)
17. Gong, X.H.: Continuity of the solution set to parametric weak vector equilibrium problems. J. Opt. Theory Appl. doi:10.1007/s10957-008-9429-8
18. Gwinner, J.: On the Penalty Method for Constrained Variational Inequalities Optimization: Theory and Algorithms, pp. 197–221. Lecture Notes in Pure and Appl. Math., vol 86. Dekker, New York (1983)
19. Gwinner, J.: Stability of monotone variational inequalities with various appplications. In: Giannessi, F., Maugeri, A. (eds.) Variational Inequalities and Network Equilibrium Problems, Plenum Press, New York (1995)
20. Gwinner, J.: Note on pseudomonotone functions, regularization, and relaxed coerciveness. Nonlinear Anal. **30**, 4217–4227 (1997)
21. Khanh, P.Q., Luu, L.M.: Upper Semicontinuity of the solution set to parametric vector quasivariational inequalities. J. Glob. Optim. **32**, 569–580 (2005)
22. Khanh, P.Q., Luu, L.M.: Lower semicontinuity and upper semicontinuity of the solution sets to parametric multivalued quasivariational inequalities. J. Opt. Theory Appl. **133**(1), 329–339 (2007)
23. Kimura, K., Yao, J.C.: Semicontinuity of Solution Mappings of Parametric Generalized Vector Equilibrium Problems. J. Opt. Theory Appl. **138**(3), 429–443 (2008)

24. Konnov, I.: Generalized monotone equilibrium problems and variational inequalities. In: Hadjisavvas, N., Komlósi, S., Schaible, S. (eds.) Handbook of Generalized Convexity and Generalized Monotonicity, vol. 76. Springer (2000)
25. Inoan, D., Kolumbán, J.: Shape design for systems governed by nonlinear hemivariational inequalities. Nonlinear Anal. Forum **7**(2), 181–197 (2002)
26. Inoan, D., Kolumbán, J.: On pseudomonotone set-valued mappings. Nonlinear Anal. **68**, 47–53 (2008)
27. Jofré, A., Wets, R.J.-B.: Continuity properties of Walras equilibrium points. Ann. Oper. Res. **114**, 229–243 (2002)
28. Leray, J., Lions, J.L.: Quelques résultats de Višik sur les problémes elliptiques non linéaires par les méthodes de Minty-Browder. Bull. Soc. Math. France **93**, 97–107 (1965)
29. Li, S.J., Chen, G.Y., Teo, K.L.: On the stability of generalized vector quasivariational inequality problems. J. Opt. Theory Appl. **107**(1), 35–50 (2000)
30. Lignola, M.B., Morgan, J.: Convergence of solutions of quasivariational inequalities and applications. Topol. Methods Nonlinear Anal. **10**, 375–385 (1997)
31. Lignola, M.B., Morgan, J.: Convergence for variational inequalities and generalized variational inequalities. Atti del Seminario Matematico E Fisico, Universitá di Modena **45**, 377–388 (1997)
32. Lignola, M.B., Morgan, J.: Generalized variational inequalities with pseudomonotone operators under perturbations. J. Opt. Theory Appl. **101**(1), 213–220 (1999)
33. Liskovets, O.L.: Regularized variational inequalities pseudo-monotone operators on approximately given domains. Diff. Eq. **25**, 1395–1401 (1989)
34. Rouhani, B.D., Khan, A.A., Raciti, F.: Penalization and regularization for multivalued pseudo-monotone variational inequalities with Mosco approximation on constraint sets. J. Glob. Optim. **40**, 147–153 (2008)
35. Lucchetti, R.: Convexity and Well-Posed Problems, CMS Books in Mathematics. Canadian Mathematical Society, Springer (2006)
36. Margiocco, M., Patrone, F., Pusillo Chicco, L.: A new approach to Tikhonov well-posedness for Nash equilibria. Optimization **40**, 385–400 (1997)
37. Margiocco, M., Patrone, F., Pusillo Chicco, L.: Metric Characterizations of Tikhonov Well-posedness in Value. J. Opt. Theory Appl. **100**(2), 377–387 (1999)
38. Mosco, U.: Convergence of convex sets and of solutions of variational inequalities. Adv. Math. **3**, 510–585 (1969)
39. Revalski, J.P.: Various aspects of well-posedness of optimization problems. In: Lucchetti, R., Revalski, J.P. (eds.) Recent Developments in Well-Posed Variational Problems, pp. 229–256. Kluwer Academic Publishers, Dordrecht (1995)
40. Showalter, R.E.: Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, vol. 49. American Math. Society (2000)
41. Tan, K.-K, Yu, J., Yuan, X.-Z.: The stability of Ky Fan's points. Proc. Am. Math. Soc. **123**, 1511-1519 (1995)
42. Yang, H., Yu, J.: Unified approaches to well-posed with some applications. J. Glob. Optim. **31**, 371–381 (2005)
43. Yu, J., Yang, H., Yu, C.: Well-posed Ky Fan's point, quasi-variational inequality and Nash equlibrium problems. Nonlinear Anal. **66**, 777–790 (2007)